Inclusion versus Recovery in Belief Base Dynamics

Renata Wassermann
Department of Computer Science
Institute of Mathematics and Statistics
University of São Paulo, Brazil
renata@ime.usp.br

Abstract

When contracting a formula from a belief base, two desiderata compete: one wants to avoid including any new belief in the process (inclusion) but may want to be able to recover information that was in the base before the contraction took place (recovery).

The AGM paradigm imposes both constraints on contraction operations. However, for finite belief bases inclusion and recovery cannot be simultaneously satisfied.

In this paper, we examine constructions that weaken the inclusion constraint and retain some form of recovery.

1 Introduction

Belief Revision [Gär88, GR95, Han99] deals with the problem of accommodating new information into a body of existing beliefs. The new piece of information may be inconsistent with the previous information held by the agent. In this case, he may have to give up some previous beliefs.

The problem has been extensively studied in the literature and most formal proposals derive from what is known as the *AGM paradigm*, due to the initials of the authors of [AGM85]. In the AGM paradigm, three operations of belief change are distinguished: expansion, which is the simple addition of a new belief; contraction, which consists in removing the desired belief; and revision, which consists in adding a new belief in such a way that the resulting set is consistent.

In the AGM paradigm, the beliefs of an agent are represented by a set of formulas closed under logical consequence, a *belief set*. The operation of expansion is obtained by adding the new belief and closing the resulting set under logical consequence. The operations of contraction and revision are not uniquely defined, but restricted by a set of desired axioms (the *rationality postulates*). Several mathematical constructions were proposed that have the property of being equivalent to the set of postulates, in the sense that not only an operation following these constructions satisfies the postulates, but also any operation satisfying the postulates can be obtained from these constructions [Gär88]. All of these constructions satisfy the Levi identity, that shows how revision can be obtained from contraction and expansion. Therefore, in this paper we will concentrate on the contraction operation.

The six basic AGM postulates for contraction are listed below:

(K-1) $K - \varphi$ is a belief set (*closure*)
(K-2) $K - \varphi \subseteq K$ (*inclusion*)
(K-3) If $\varphi \notin K$, then $K - \varphi = K$ (*vacuity*)
(K-4) If not $\vdash \varphi$, then $\varphi \notin K - \varphi$ (*success*)
(K-5) $K \subseteq (K - \varphi) + \varphi$ (*recovery*)
(K-6) If $\vdash \varphi \leftrightarrow \psi$, then $K - \varphi = K - \psi$ (*extensionality*)

These postulates are supposed to capture the intuition behind the operation of giving up a belief in a rational way. Postulate **(K-1)** says that the result of contracting a belief set by a formula should again be a belief set. The next postulate assures that in an operation of contraction no new formulas are added to the initial belief set. If the formula to be contracted is not an element of the initial belief set, then by **(K-3)** nothing changes. Postulate **(K-4)** says that unless the sentence to be contracted is logically valid (and hence, an element of every theory), it is not an element of the resulting belief set. The recovery postulate **(K-5)** is the most controversial one [Mak87]. It says that a contraction should be recoverable, that is, that the original belief set should be recovered by expanding by the formula that was contracted. The last postulate assures that contraction by logically equivalent sentences produces the same output.

The postulate of *recovery* has been debated since the very beginning of the AGM paradigm. We want to avoid contraction operations that simply discard all the beliefs. Intuitively, when we contract by a formula α, we want to discard a minimal subset of the belief set such that the resulting set does not contain α. This is known as the Principle of Minimal Change. Recovery is one way to try to capture this minimality, but some examples show that the postulate may be too strong:

Example [Han99]: "I have read in a book about Cleopatra that she had both a son and a daughter. My set of beliefs therefore contains both

p and q, where p denotes that Cleopatra had a son and q that she had a daughter. I then learn from a knowledgeable friend that the book is in fact a historical novel. After that I contract $p \vee q$ from my set of beliefs, i.e., I do not any longer believe that Cleopatra had a child. Soon after that, however, I learn from a reliable source that Cleopatra had a child. It seems perfectly reasonable for me to then add $p \vee q$ to my set of beliefs without also reintroducing either p or q. This contradicts Recovery."

There are several constructions in the literature that do not satisfy recovery [Mak87, Fer01], but for belief sets they all present some other undesirable property.

An alternative representation to belief sets is the use of belief bases, i.e., sets of formulas that are not necessarily closed under logical consequence. Besides being more expressive (as we can always derive the corresponding belief set taking the closure of a belief base), belief bases have clear advantages from the computational point of view. For belief bases, there are constructions that seem reasonable and do not satisfy recovery, as we will show later.

The postulates of *success* and *inclusion* are sometimes seen as the minimal requirements for a contraction operation. Together they state that the desired belief is removed and nothing else new is included in the belief set.

In this paper we will discuss the role and adequacy of the inclusion postulate. If we move from closed belief sets to belief bases, the postulate may be too restrictive, as the following example shows:

Example: Suppose we have a belief base containing $p \wedge q$, which stands for the fact that Cleopatra had a son (p) and a that she had a daughter (q). If we want to contract the base by p, i.e., we want to remove the belief that she had a son, we have to give up the whole conjunction, and since the formula q is not included in the base, we give up the belief that Cleopatra had a daughter too.

We would like to have a construction that allows us to keep parts of formulas being removed. In this paper, we will present some ideas on such constructions.

The rest of the paper is organized as follows: in the next Section, we introduce partial meet contraction of belief bases and its properties. Then we give an example of contraction without inclusion and propose a more general construction. We show that this construction can be instantiated and give rise to operations with different properties.

Throughout the paper we consider Cn to be a Tarskian consequence operator (i.e., satisfying inclusion, monotony and idempotence). We use lower case letters to denote atoms, Greek lower case for formulas and upper case letters to denote set of formulas.

2 Contraction of Belief Bases

In this section, we will present postulates and a construction for contraction of belief bases. We will then discuss the properties of the operation.

The first construction for contraction that was proposed and proved to be equivalent to the six AGM postulates was Partial Meet Contraction [AGM85]. The operation is based on the idea of selecting some of the maximal subsets of a belief set that do not imply the formula to be contracted and taking their intersection. The same construction can be used for belief bases:

Definition 2.1 *[AM82] Let X be a set of formulas and α a formula. For any set Y, $Y \in X \perp \alpha$ if and only if:*

- $Y \subseteq X$
- $Y \not\vdash \alpha$
- *For all Y' such that $Y \subset Y' \subseteq X$, $Y' \vdash \alpha$.*

Definition 2.2 *[AGM85] A* **selection function** *for X is a function γ such that:*

- *If $X \perp \alpha \neq \emptyset$, then $\emptyset \neq \gamma(X \perp \alpha) \subseteq X \perp \alpha$.*
- *Otherwise, $\gamma(X \perp \alpha) = \{X\}$.*

Definition 2.3 *[AGM85, Han91b] The* **partial meet base contraction operator** *on B based on a selection function γ is the operator $\dot{-}_\gamma$ such that for all sentences α:*
$B \dot{-}_\gamma \alpha = \bigcap \gamma(B \perp \alpha)$.

It is easy to see that $\dot{-}_\gamma$ satisfies **(K-2)**, **(K-3)**, **(K-4)**, and **(K-6)**. To see that it does not satisfy **(K-5)**, we can look at the Cleopatra example:
$B = \{p, q\}, B \perp (p \vee q) = \{\emptyset\}, \gamma(B \perp (p \vee q)) = \{\emptyset\}, B \dot{-}_\gamma (p \vee q) = \emptyset$.
But $B \not\subseteq B \dot{-}_\gamma (p \vee q) + (p \vee q) = \{p \vee q\}$.

Stronger than that, even if we consider an operation on belief sets generated from $\dot{-}_\gamma$, the example shows that it does not satisfy recovery, i.e.,

$Cn(B) \not\subseteq Cn(B \dot{-}_\gamma (p \vee q) + (p \vee q) = \{p \vee q\})$.

Actually, Hansson [Han99] has shown that under very general conditions, any base-generated contraction operation fails to satisfy recovery.

Hansson has proposed an alternative axiomatization for partial meet base contraction and proven that it is equivalent to the construction:

Theorem 2.4 *[Han92] An operator $\dot{-}$ is an operator of partial meet base contraction on B if and only if:*

- *If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(B \dot{-} \alpha)$ (success)*
- *$B \dot{-} \alpha \subseteq B$ (inclusion)*
- *If $\beta \in B \setminus (B \dot{-} \alpha)$, then there is some B' such that $B \dot{-} \alpha \subseteq B' \subseteq B$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$ (relevance)*
- *If for all subsets B' of B, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B \dot{-} \alpha = B \dot{-} \beta$ (uniformity)*

The last postulate (uniformity) is a stronger version of extensionality. Instead of recovery, Hansson suggested the relevance postulate in order to capture the idea of minimal change. In a contraction by α, the only formulas given up are those that somehow contribute to the derivation of α.

As we have seen, adapting the traditional AGM construction of partial meet contraction to belief bases results in an operation that satisfies inclusion and success, but not recovery.

In the next section, we will see how weakening inclusion can bring back some sort of recovery.

3 Contraction without Inclusion

We have seen in the Introduction that even the inclusion postulate may be too strong when talking about belief bases. We would like sometimes to retain parts of the beliefs that are being given up, which means replacing them by some of their consequences.

The second Cleopatra example shows that we may want to have $\{p \wedge q\} - p = \{q\}$, i.e., instead of giving up the conjunction, replace it by one of its conjuncts.

Giving up inclusion all together may bring too much freedom for the allowed constructions: we could end up adding just any formula that did not threaten success, such as having $\{p\} - p = \{q\}$. The idea is to weaken the postulate so as to allow the addition of some kinds of formulas.

Hansson [Han89] has proposed a weaker version of inclusion that he called *logical inclusion*:

$Cn(B - \alpha) \subseteq Cn(B)$

Note that if Cn is Tarskian, logical inclusion is equivalent to

$B - \alpha \subseteq Cn(B)$

We will use any of the two forms in this paper.

Hansson called an operation satisfying success and logical inclusion a *pseudo-contraction*.

Following the same line, we may think of a weaker version of recovery:

$B \subseteq Cn(B - \alpha + \alpha)$ (logical recovery)

Nebel [Neb89] has proposed a construction of pseudo-contraction that satisfies logical recovery:

Definition 3.1 *[Neb89] Let B be a finite belief base, α a formula and γ a selection function.*

$$B - \alpha = \begin{cases} B & \textit{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma(B \bot \alpha) \cup \{\alpha \to \beta | \beta \in B\} & \textit{otherwise} \end{cases}$$

The idea is to add to the partial meet contraction some consequences of the formulas of the original belief base so that they allow for the recovery of the contracted sentences.

It is easy to see that not all formulas of the form $\alpha \to \beta$ have to be added, we can restrict ourselves to those β such that $\beta \in B \setminus \bigcap \gamma(B \bot \alpha)$:

Proposition 3.2 *Let B be a finite belief base, α a formula and γ a selection function and define the contraction $B - \alpha$ as:*

$$B - \alpha = \begin{cases} B & \textit{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma(B \bot \alpha) \cup \{\alpha \to \beta | \beta \in B \setminus \bigcap \gamma(B \bot \alpha)\} & \textit{otherwise} \end{cases}$$

Then $-$ satisfies logical inclusion, vacuity (if $\alpha \notin Cn(B)$, then $B - \alpha = B$), success, logical recovery and extensionality.

Proof: Logical inclusion, vacuity, logical recovery and extensionality follow directly from the construction. To see that success is satisfied, suppose that $\alpha \notin Cn(\emptyset)$ and $\alpha \in Cn(B \dot{-} \alpha)$. Then there are $\beta_1, \beta_2, ..., \beta_n$ in $B \setminus \bigcap \gamma(B \bot \alpha)$ such that $\bigcap \gamma(B \bot \alpha) \cup \{\alpha \to \beta_1, \alpha \to \beta_2, ..., \alpha \to \beta_n\} \vdash \alpha$. Using the deduction theorem and the fact that $(\alpha \to \beta) \to \alpha$ is equivalent to α, we have that $\bigcap \gamma(B \bot \alpha) \vdash \alpha$, which we know cannot be the case (since this is the traditional partial meet construction). $\square$

In a sense, Nebel's construction seems to have been coined to satisfy recovery. There is no other intuition about why exactly the implications of the form $\{\alpha \rightarrow \beta | \beta \in B\}$ should be added.

If we go back to our second Cleopatra example, Nebel's pseudo-contraction (or the more economic form of it) would make $\{p \wedge q\} - p = \{p \rightarrow p \wedge q\}$, which means that if we do not believe anymore that Cleopatra had a son, we do not know anything about her having a daughter.

We will propose here a slightly different construction. Suppose that we have a partial meet contraction for a belief base B, with an associated selection function γ.

Let Cn^* denote an operation that generates some consequences of a set of formulas, i.e., $Cn^*(A) \subseteq Cn(A)$. As an example, we could have $Cn^*(A) = \{\alpha \vee \beta | \alpha \in A\}$.

Definition 3.3 *Let γ be a selection function for B and let B^* contain B. An extension of γ to B^* is a selection function γ^* such that for every $Y \in \gamma^*(B^* \perp \alpha)$ there is an $X \in \gamma(B \perp \alpha)$ such that $X \subseteq Y$.*

Observation 3.4 *Let γ be a selection function for B and let B^* contain B. If γ^* is an extension of γ to B^*, then for any $X \in \gamma(B \perp \alpha)$ there is a $Y \in \gamma^*(B^* \perp \alpha)$ such that $X \subseteq Y$.*

Definition 3.5 *Let B be a finite belief base, α a formula and γ a selection function for B. The general partial meet pseudo-contraction $B - \alpha$ is given by:*

$$B - \alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma^*(B^* \perp \alpha) & \text{otherwise} \end{cases}$$

where $B^* = B \cup Cn^*(\{\beta | \beta \in B \setminus \bigcap \gamma(B \perp \alpha)\})$ and γ^* is an extension of γ.

Proposition 3.6 *The pseudo-contraction operation defined as above satisfies logical inclusion, vacuity ($\alpha \notin Cn(B)$, then $B - \alpha = B$), success, and extensionality.*

Proof: Directly from the definition and the observation that $\bigcap \gamma(B \perp \alpha)$ satisfies extensionality, thus equivalent formulas generate the same B^*. □

Whether this construction satisfies logical recovery or not depends on the Cn^* used. It is easy to see that the restricted form of Nebel's pseudo-contraction can be obtained if we take $Cn^*(A) = \{\alpha \rightarrow \beta | \beta \in A\}$. In this particular case, we do have logical recovery.

We can think of different definitions for Cn^* depending on the intuitions. For example, if we look once more to the Cleopatra example, we may want to allow the addition of only those consequences which are subformulas of the formulas removed. The pseudo-contraction operation using this definition of Cn^* does not satisfy logical recovery, as can be seen from the following example:

Let $B = \{p, p \to q\}$. Then $B \perp q = \{\{p\}, \{p \to q\}\}$. Suppose that $\gamma(B \perp q) = \{\{p \to q\}\}$. Since the only consequence of $p \to q$ that is a subformula of it is the whole formula, we have $B^* = B$. And since γ^* extends γ, we must have $\gamma^*(B \perp q) = \{\{p \to q\}\}$. Hence $B - q = \{p \to q\}$ and $B \not\subseteq Cn(B - q + q)$.

The operation does not even satisfy the weaker postulate of relevance, as inclusion may not hold and thus we may not have a set B^* such that $B \dot{-} \alpha \subseteq B^* \subseteq B$. It does however satisfy a still weaker version of relevance, introduced by Hansson in [Han91a]:

(core-retainment) If $\beta \in B \setminus (B \dot{-} \alpha)$, then there is some B' such that $B' \subseteq B$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$

In fact, core-retainment is satisfied for any general partial meet pseudo-contraction, regardless of the particular Cn^* used (we do not even need to have $Cn^*(A) \subseteq Cn(A)$):

Proposition 3.7 *For any operator Cn^*, general partial meet pseudo-contraction satisfies core-retainment.*

Proof: Let $\beta \in B \setminus (B \dot{-} \alpha)$. Then, there is $X \in \gamma^*(B^* \perp \alpha)$ such that $\beta \notin X$. As γ^* extends γ, there must be $Y \in \gamma(B \perp \alpha)$ such that $Y \subseteq X$. Then take $B' = Y$. □

Another way to relax relevance is to follow what was done with inclusion and recovery:

(logical relevance) If $\beta \in B \setminus (B \dot{-} \alpha)$, then there is some B' such that $B \dot{-} \alpha \subseteq B' \subseteq Cn(B)$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$.

Proposition 3.8 *If for every set A, $Cn^*(A) \subseteq Cn(A)$, then general partial meet pseudo-contraction satisfies logical relevance.*

Proof: Let $\beta \in B \setminus (B \dot{-} \alpha)$. Then, there is $X \in \gamma^*(B^* \perp \alpha)$ such that $\beta \notin X$. Since $B^* \subseteq Cn(B)$, we can make $B' = X$. □

4 Conclusions and Future Work

In this paper, we have discussed how weakening the inclusion requirement can provide some sort of recovery for operations on belief bases.

We have proposed a general construction based on partial meet that allows some consequences of the original belief base to be added during contraction. We have shown that Nebel's construction [Neb89] is a special case of our proposal. We have then shown that our proposal allows for operations that satisfy recovery (like Nebel's), but also operations that only satisfy weaker versions of it.

We are left with a whole spectrum of operations varying from those where any formula can be added (i.e., no inclusion postulate) to those where nothing can be added. In between, when we use a weaker notion of inclusion, there are constructions satisfying recovery in different degrees.

Future work includes studying other possible construction and proving representation results for them.

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